Journal of Mathematics,
Osaka City University,
Vol. 12, No. 1-2.

On randomizations of confidence intervals

By Tokitake KUSAMA

(Received September 20, 1961)

In recent years it seems to the author that it has been devoted by some statisticians and mathematicians to rewrite a considerable part of the theory of statistics in the terms of statistical decision functions. The theory of confidence intervals is not exceptional. This leads writers of the theory to the consideration of randomized confidence intervals for the sake of the theoretical and practical treatment of confidence intervals.

The author has been informed, up to this time, of two types of expressions of randomized confidence intervals.

(1) W-type randomization; depending on every sample point x observed, we choose a random machine m_x and then select an interval by means of m_x. For example, we shoot, for every observed x, an arrow which hits in a subset S of the half plane $A=\{(u, v)\,|\,u\leqq v\}$ with probability $m_x(S)$. If it hits a point $a=(u, v)$, we estimate the true parameter θ to belong to the interval limited by u and v.

The readers can see such a randomization in Wald's Book (Wald [1]). We shall call this a W-type randomization.

(2) S-type randomization; some writers have used another type of randomization though it is rather vaguely defined (Stein [2]). It gives only the probability $\varphi_x(\theta)$ that confidence interval contains θ for observed x. So it does not tell us how to estimate the true parameter θ. We shall call it an S-type randomization. There is some φ_x which can not be expressed in the preceding form.

The first aim of this note is to give necessary and sufficient conditions for an S-type to be expressible as a W-type. According to them, φ_x has an expression for a W-type if and only if the total variation of $\varphi_x(\theta)$ as a function of θ does not exceed 2 for every x.

Next we consider the set $\mathfrak{F}^*$ of all φ_x whose total variation as a function of θ is exactly 2 for every x and the set $\mathfrak{M}^*$ of all m_x by which we estimate, for every x, an interval containing θ to be surely on a curve c_x in A. And we shall prove that these subclasses $\mathfrak{F}^*$ and $\mathfrak{M}^*$ are essentially complete in the class of all W-type randomizations and of all S-type randomizations respectively.

At last we shall show equivalences of these classes. These equivalences are not so clear as at the first glance when we adopt the confidence coefficient and the average length as a criterion of optimalities.

In §1, we introduce concepts of randomized intervals, and prove some lemmas. In §2 we define confidence intervals as mappings of sample x into randomized intervals introduced in §1 and prove essential completeness of subclasses above mentioned and equivalences of both types under two risks, the confidence coefficient and the average length.

§1. Lemmas. We define M, Φ and Φ^* as follows.

M; the set of all probability measures on the half plane $A=\{(u, v)\mid u\leqq v\}$.

Φ ; the set of all upper semicontinuous functions $\varphi(\theta)$ defined on $(-\infty, +\infty)$ satisfying

$$(\alpha)\quad 0\leqq\varphi(\theta)\leqq 1,$$
$$(\beta)\quad \lim_{\theta\to\pm\infty}\varphi(\theta)=0,$$
$$(\gamma)\quad \mathrm{Var}\,\varphi(\theta)\leqq 2,$$

where $\mathrm{Var}\, f(\theta)$ means the total variation of $f(\theta)$ on $(-\infty, +\infty)$ as a function of θ.

Φ^*; the set of all φ^* in Φ such that

$$(\gamma')\quad \mathrm{Var}\,\varphi^*(\theta)=2.$$

LEMMA 1. For a given $\varphi(\in\Phi)$ and a given θ_0, there exists φ^* in Φ^* satisfying $\varphi(\theta)=\varphi^*(\theta)$ for all $\theta(\neq\theta_0)$ and $\varphi(\theta_0)\leqq\varphi^*(\theta_0)$. Hence we have $\int_{-\infty}^{+\infty}\varphi^*(\theta)d\theta=\int_{-\infty}^{+\infty}\varphi(\theta)d\theta$.

PROOF. It is sufficient to prove this in the case where $\mathrm{Var}\,\varphi(\theta)<2$. Put φ^* as follows,

$$\varphi^*(\theta)=\varphi(\theta)+1-\mathrm{Var}\,\varphi(\theta)/2 \qquad \text{for}\quad \theta=\theta_0,$$
$$\varphi^*(\theta)=\varphi(\theta) \qquad \text{for}\quad \theta\neq\theta_0.$$

It is clear $0\leqq\varphi^*(\theta)$ since $1-\mathrm{Var}\,\varphi(\theta)/2>0$. We shall show $\varphi^*(\theta)\leqq 1$. We may assume without any loss of generality that $\varphi(\theta_0)>0$. By the assumption (β) of φ, for any given positive $\varepsilon<\varphi(\theta_0)$, we can choose η_1, η_2 such that $\eta_1<\theta_0<\eta_2$ and $\varphi(\theta_0)-\varphi(\eta_i)>\varphi(\theta_0)-\varepsilon$ $(i=1, 2)$. So we have

$$\mathrm{Var}\,\varphi(\theta)\geqq\sum_{i=1}^{2}(\varphi(\theta_0)-\varphi(\eta_i))>2\varphi(\theta_0)-2\varepsilon,$$

which shows $\mathrm{Var}\,\varphi(\theta)/2\geqq\varphi(\theta_0)$ and consequently $1\geqq\varphi^*(\theta_0)$. So φ^* satisfies the condition (α).

Next we shall show φ^* satisfies (γ'). Suppose that $\mathrm{Var}\,\varphi^*(\theta)<2$. For any given ε such that $0<\varepsilon<1-\mathrm{Var}\,\varphi(\theta)/2$, there exist n reals $\eta_1, \eta_2, \cdots, \eta_n$ such that

$$(1)\qquad -\infty<\eta_1<\eta_2<\cdots\cdots<\eta_n<+\infty,$$

$$(2)\qquad \sum_{i=2}^{n}|\varphi(\eta_i)-\varphi(\eta_{i-1})|>\mathrm{Var}\,\varphi(\theta)-\varepsilon.$$

As the left hand side of (2) does not decrease by adding a point to $\{\eta_i;\, i=1, 2,$

$\cdots, n\}$, we can assume that one of η_i $(1<i<n)$, say η_s, coincides with θ_0. And by the upper semicontinuity of φ, we can choose $\eta_{s\pm1}$ sufficiently near η_s such that

$$\varphi(\eta_s)>\varphi(\eta_{s\pm1})-\varepsilon/2\,. \tag{3}$$

By (3) and $0<\varepsilon<1-\mathrm{Var}\,\varphi(\theta)/2$, we have

$$\begin{aligned}\varphi^*(\eta_s)-\varphi^*(\eta_{s\pm1}) &= \varphi(\eta_s)+1-\mathrm{Var}\,\varphi(\theta)/2-\varphi(\eta_{s\pm1})\\ &> -\varepsilon/2+1-\mathrm{Var}\,\varphi(\theta)/2>0\,.\end{aligned}$$

So, if $\varphi(\eta_s)\geqq\varphi(\eta_{s\pm1})$, we have

$$\begin{aligned}&|\varphi^*(\eta_s)-\varphi^*(\eta_{s\pm1})|-|\varphi(\eta_s)-\varphi(\eta_{s\pm1})|\\ &\quad=\varphi(\eta_s)+1-\mathrm{Var}\,\varphi(\theta)/2-\varphi(\eta_{s\pm1})-\varphi(\eta_s)+\varphi(\eta_{s\pm1})\\ &\quad=1-\mathrm{Var}\,\varphi(\theta)/2\,.\end{aligned}$$

If $\varphi(\eta_s)<\varphi(\eta_{s\pm1})$, we have

$$\begin{aligned}&|\varphi^*(\eta_s)-\varphi^*(\eta_{s\pm1})|-|\varphi(\eta_s)-\varphi(\eta_{s\pm1})|\\ &\quad=\varphi(\eta_s)+1-\mathrm{Var}\,\varphi(\theta)/2-\varphi(\eta_{s\pm1})-\varphi(\eta_{s\pm1})+\varphi(\eta_s)\\ &\quad=1-\mathrm{Var}\,\varphi(\theta)/2+2(\varphi(\eta_s)-\varphi(\eta_{s\pm1}))>1-\mathrm{Var}\,\varphi(\theta)/2-\varepsilon\,.\end{aligned}$$

In all cases, we have

$$|\varphi^*(\eta_s)-\varphi^*(\eta_{s\pm1})|-|\varphi(\eta_s)-\varphi(\eta_{s\pm1})|>1-\mathrm{Var}\,\varphi(\theta)/2-\varepsilon$$

and then

$$\begin{aligned}\sum_{i=2}^{n}|\varphi^*(\eta_i)-\varphi^*(\eta_{i-1})|&>\sum_{i=2}^{n}|\varphi(\eta_i)-\varphi(\eta_{i-1})|+2-\mathrm{Var}\,\varphi(\theta)-2\varepsilon\\ &>\mathrm{Var}\,\varphi(\theta)-\varepsilon+2-\mathrm{Var}\,\varphi(\theta)-2\varepsilon=2-3\varepsilon\,.\end{aligned}$$

Therefore $\mathrm{Var}\,\varphi^*(\theta)\geqq2$ holds.

Suppose that $\mathrm{Var}\,\varphi^*(\theta)>2$. Then there exist $\eta_1, \eta_2, \cdots, \eta_n$ such that $-\infty<\eta_1<\cdots\cdots<\eta_n<+\infty$ and $\sum_{i=2}^{n}|\varphi^*(\eta_i)-\varphi^*(\eta_{i-1})|>2$. Here an η_i, say η_s, coincides clearly with θ_0. Thus we have

$$\begin{aligned}2&<\sum_{i=2}^{s-1}+\sum_{i=s+2}^{n}+|\varphi^*(\eta_s)-\varphi^*(\eta_{s-1})|+|\varphi^*(\eta_{s+1})-\varphi^*(\eta_s)|\\ &\leqq\sum_{i=2}^{s-1}+\sum_{i=s+2}^{n}+|\varphi(\eta_s)-\varphi(\eta_{s-1})|+|\varphi(\eta_{s+1})-\varphi(\eta_s)|+1\\ &\qquad\qquad-\mathrm{Var}\,\varphi(\theta)/2+1-\mathrm{Var}\,\varphi(\theta)/2\,.\end{aligned}$$

Consequently we have

$$\mathrm{Var}\,\varphi(\theta)<\sum_{i=2}^{n}|\varphi(\eta_i)-\varphi(\eta_{i-1})|$$

which is a contradiction. So φ^* satisfies (γ').

It is easy to prove φ^* satisfies (β). The upper semi-continuity of φ^* is assured by definition of φ^*.

LEMMA 2. If we put $\varphi(\theta)=m(S_\theta)$ for any $m(\in \overset{*}{M})$, then we have $\varphi\in\Phi$ and

$$\int_{-\infty}^{+\infty}\varphi(\theta)\,d\theta=\int_A (v-u)\,dm(u,\ v)\ ,$$

where S_θ denotes the set $\{(u, v)\,|\,u\leqq\theta\leqq v\}$ of A.

PROOF. $\varphi(\theta)$ is an upper semicontinuous function of θ since S_θ is closed. It is clear φ satisfies (α), (β). To show (γ) for φ, suppose that $\mathrm{Var}\,\varphi(\theta)>2$. Then there exist $\theta_1, \theta_2, \cdots, \theta_{2m}$ $(-\infty<\theta_1<\theta_2\leqq\theta_3<\theta_4\cdots\cdots<\theta_{2m}<+\infty)$ such that either

$$(1)\qquad \sum_{k=1}^{m}(\varphi(\theta_{2k})-\varphi(\theta_{2k-1}))>1 \qquad (\varphi(\theta_{2k})>\varphi(\theta_{2k-1}))$$

or

$$(2)\qquad \sum_{k=1}^{m}(\varphi(\theta_{2k-1})-\varphi(\theta_{2k}))>1 \qquad (\varphi(\theta_{2k-1})>\varphi(\theta_{2k}))$$

holds. If (1) is the case, we have

$$\begin{aligned} m(A) &\geqq m(S_{\theta_1})+\sum_{k=1}^{m} m(S_{\theta_{2k}}-S_{\theta_{2k-1}}) \\ &\geqq \varphi(\theta_1)+\sum_{k=1}^{m}(\varphi(\theta_{2k})-\varphi(\theta_{2k-1}))>1 \end{aligned}$$

which is a contradiction. On the other hand, we can show a quite similar contradiction for (2). Thus holds (γ).

We have

$$\int_{-\infty}^{+\infty}\varphi(\theta)\,d\theta=\int_{-\infty}^{+\infty}m(S_\theta)\,d\theta=\int_{-\infty}^{+\infty}\int_{S_\theta}dm(u,\ v)d\theta\,.$$

The last integral is equivalent to $\int_A\left(\int_u^v d\theta\right)dm\,(u, v)$, since $\{(u, v, \theta)\,|\,-\infty<\theta<\infty,$ $(u, v)\in S_\theta\}=\{(u, v, \theta)\,|\,-\infty<\theta<\infty,\ u\leqq\theta\leqq v\}=\{(u, v, \theta)\,|\,-\infty<u\leqq v<\infty,\ u\leqq\theta\leqq v\}$. Consequently we have

$$\int_{-\infty}^{\infty}\varphi(\theta)\,d\theta=\int_A\left(\int_u^v d\theta\right)dm(u,\ v)=\int_A (v-u)\,dm(u,\ v)\,.$$

Thus we complete the proof.

LEMMA 3. For a given $\varphi^*(\in\Phi^*)$, there exist monotone nonincreasing functions $u(z)$ and $v(z)$ defined on $(0, 1)$, satisfying

$$u(z)\leqq v(z) \qquad \text{for all}\quad z(\in(0,\ 1))\,,$$

$$\varphi^*(\theta)=\int_{u(z)\leqq\theta\leqq v(z)}dz\,,$$

and

$$\int_{-\infty}^{+\infty}\varphi^*(\theta)\,d\theta=\int_0^1 (v(z)-u(z))\,dz\,.$$

PROOF. By the assumption (γ') of φ^*, $\varphi^*(\theta)$ is expressed as the difference $\varphi^*(\theta)=P(\theta)-N(\theta)$, where $P(\theta)$ and $N(\theta)$ are $\underset{(-\infty,\,\theta]}{\mathrm{Var}^+}\varphi^*(\theta)$ and $\underset{(-\infty,\,\theta]}{\mathrm{Var}^-}\varphi^*(\theta)$ respectively.

By the assumptions (β) and (γ'), we have

$$\operatorname*{Lim}_{\theta\to+\infty} P(\theta) = 1\,, \qquad \operatorname*{Lim}_{\theta\to+\infty} N(\theta) = 1\,,$$

and

$$\operatorname*{Lim}_{\theta\to-\infty} P(\theta) = 0\,, \qquad \operatorname*{Lim}_{\theta\to-\infty} N(\theta) = 0\,.$$

$\varphi^*(\theta)\geqq 0$ shows that $N(\theta)\leqq P(\theta)$ for all θ, and since P and N are monotone nondecreasing functions of θ, we have $0\leqq N(\theta)\leqq P(\theta)\leqq 1$. Put $u(z)=\inf\{\theta\,|\,P(\theta)\geqq z\}$ and $v(z)=\sup\{\theta\,|\,N(\theta)\leqq z\}$ for $z\in(0,1)$. Suppose that there exists z_0 such that $u(z_0)>v(z_0)$. Then $u(z_0)>\theta>v(z_0)$ implies $P(\theta)<z_0<N(\theta)$ which is a contradiction. So we have $u(z)\leqq v(z)$ for all $z(\in(0,1))$.

It is clear from the definitions of u and v that the functions u and v of z are monotone nonincreasing.

Moreover we have

$$\varphi^*(\theta) = P(\theta)-N(\theta) = \int_{P(\theta)\geqq z\geqq N(\theta)} dz = \int_{u(z)\leqq\theta\leqq v(z)} dz$$

and

$$\int_0^1 \{v(z)-u(z)\}\,dz = \int_0^1\left(\int_{u(z)}^{v(z)} d\theta\right)dz = \int_{-\infty}^{+\infty}\int_{N(\theta)}^{P(\theta)} dz\,d\theta = \int_{-\infty}^{+\infty}\varphi^*(\theta)\,d\theta\,.$$

This completes the proof.

Lemma 4. Let $u(z)$ and $v(z)$ be monotone nonincreasing functions defined on $(0,1)$ such that $u(z)\leqq v(z)$ for all $z(\in(0,1))$. If we put $m(S)=\int_{S\ni(u(z),v(z))} dz$, then we have $m\in M$ and

$$m(S_\theta) = \int_{u(z)\leqq\theta\leqq v(z)} dz$$

and

$$\int_0^1\{v(z)-u(z)\}\,dz = \int_A (v-u)\,dm(u,\,v)\,.$$

Proof. It is obvious that $m\in M$. Since $u(z)\leqq\theta\leqq v(z)$ is equivalent to $(u(z),v(z))\in S_\theta$, we have

$$m(S_\theta) = \int_{S_\theta\ni(u(z),\,v(z))} dz = \int_{u(z)\leqq\theta\leqq v(z)} dz$$

and

$$\int_0^1\{v(z)-u(z)\}\,dz = \int_0^1\left\{\int_{u(z)\leqq\theta\leqq v(z)} d\theta\right\}dz = \int_0^1\left\{\int_{S_\theta\ni(u(z),\,v(z))} d\theta\right\}dz$$
$$= \int_{-\infty}^{\infty}\left\{\int_{S_\theta\ni(u(z),\,v(z))} dz\right\}d\theta = \int_{-\infty}^{\infty} m(S_\theta)\,d\theta = \int_{-\infty}^{\infty}\int_{S_\theta} dm(u,\,v)\,d\theta\,.$$

The last integral is equivalent to $\int_A\left(\int_u^v d\theta\right)dm(u,v)$ by the same reason as we met in the proof of the Lemma 2.

Thus we have

$$\int_0^1 \{v(z)-u(z)\}dz = \int_A \left(\int_u^v d\theta\right) dm(u, v) = \int_A (v-u)\, dm(u, v)\,,$$

which completes the proof.

§2. **Theorems.** Let (X, B) be an arbitrary measurable space where B is a σ-field of subsets of X. We assume that $\{a\} \in B$ for any $a(\in X)$, and that there exists a one to one correspondence ψ between X and some subset of $(-\infty, +\infty)$.

(1) Consider a function m_x defined on X with its range M satisfying

(a) $m_x(S_\theta)$ is a measurable function of x for every fixed θ,

(b) $\int_{-\infty}^{\infty} m_x(S_\theta)d\theta$ is a measurable function of x.

Such a function will be called a *W-type randomization* of confidence interval. The set of all W-type randomizations will be denoted by $\mathfrak{M}$.

Let C be the set of all curves of A that are expressed as $\{(u(z), v(z)) \mid z \in (0, 1)\}$, where $u(z)$ and $v(z)$ are monotone nonincreasing functions defined on $(0, 1)$. We shall denote by $\mathfrak{M}^*$ the set of all $m(\in \mathfrak{M})$ such that, for every x, there exists $c_x \in C$ satisfying $m_x(c_x)=1$.

(2) Consider a function φ_x defined on X with its range Φ satisfying

(a′) $\varphi_x(\theta)$ is a measurable function of x for every fixed θ,

(b′) $\int_{-\infty}^{\infty} \varphi_x(\theta)d\theta$ is a measurable function of x.

It will be called an *S-type randomization* of confidence interval. We shall denote by $\mathfrak{F}$ the set of all S-type randomizations.

Moreover we shall denote by $\mathfrak{F}^*$ the set of all φ^* satisfying $\varphi_x^* \in \Phi^*$ for all x.

Let $G=\{p_\theta \mid \theta \in (-\infty, +\infty)\}$ be a family of probability measures on (X, B) with a real parameter $\theta \in (-\infty, +\infty)$ and we define $w_1(\theta, m)$, $c_1(\theta, m)$, $w_2(\theta, \varphi)$ and $c_2(\theta, \varphi)$ as follows;

$$w_1(\theta, m) = \int_X \int_A (v-u)\, dm_x(u, v)\, dp_\theta(x)\,,$$

$$c_1(\theta, m) = \int_X m_x(S_\theta)\, dp_\theta(x)\,,$$

$$w_2(\theta, \varphi) = \int_X \int_{-\infty}^{+\infty} \varphi_x(\eta)\, d\eta dp_\theta(x)\,,$$

and

$$c_2(\theta, \varphi) = \int_X \varphi_x(\theta)\, dp_\theta(x)\,.$$

These integrals are clearly legitimate by definitions of m and φ except $w_1(\theta, m)$. But Lemma 2 shows $\int_{-\infty}^{+\infty} m_x(S_\theta)\, d\theta = \int_A (v-u)\, dm_x(u, v)$ and hence $w_1(\theta, m)$ can be also defined by (b).

A subset $D \subset \mathfrak{M}$ $(\subset \mathfrak{F})$ is said to be essentially complete in a subset $E \subset \mathfrak{M}$ $(\subset \mathfrak{F})$, if, for a given $m \in E$ $(\varphi \in E)$, there exists $m' \in D$ $(\varphi' \in D)$ satisfying $w_1(\theta, m) \geqq w_1(\theta, m')$ and $c_1(\theta, m') \geqq c_1(\theta, m)$ for all $\theta \in \Omega$ $(w_2(\theta, \varphi) \geqq w_2(\theta, \varphi')$ and $c_2(\theta, \varphi') \geqq c_2(\theta, \varphi)$ for all $\theta \in \Omega)$.

A subset $D \subset \mathfrak{M}$ $(\subset \mathfrak{F})$ is also said to be essentially complete in a subset $E \subset \mathfrak{F}$ $(\subset \mathfrak{M})$, if, for a given $\varphi \in E$ $(m \in E)$, there exists $m \in D$ $(\varphi \in D)$ such that $w_2(\theta, \varphi) \geqq w_1(\theta, m)$ and $c_1(\theta, m) \geqq c_2(\theta, \varphi)$ for all $\theta \in \Omega$ $(w_1(\theta, m) \geqq w_2(\theta, \varphi)$ and $c_2(\theta, \varphi) \geqq c_1(\theta, m)$ for all $\theta \in \Omega)$.

Two sets D and E are said to be equivalent if D is essentially complete in E and E is essentially complete in D. When D is essentially complete in E, we write $D \prec E$.

THEOREM 1. *$\mathfrak{F}^*$ is essentially complete in $\mathfrak{F}$.*

PROOF. By Lemma 1, for a given $\varphi \in \mathfrak{F}$, there exists $\varphi_x^* \in \Phi^*$ for every x such that $\varphi_x(\theta) = \varphi_x^*(\theta)$ for $\theta (\neq \psi(x))$, $\varphi_x(\theta) \leqq \varphi_x^*(\theta)$ for $\theta = \psi(x)$ and hence $\int_{-\infty}^{\infty} \varphi_x(\theta) d\theta = \int_{-\infty}^{\infty} \varphi_x^*(\theta)\, d\theta$.

To show $\varphi^* \in \mathfrak{F}^*$, it remains only to prove measurability of $\varphi_x^*(\theta)$ with respect to x. But by the definition of ψ, $\varphi_x^*(\theta) = \varphi_x(\theta)$ except at most one point for every fixed θ. So by the assumption on σ-field B, $\varphi_x^*(\theta)$ is measurable with respect to x.

It is clear that $w_2(\theta, \varphi) = w_2(\theta, \varphi^*)$ and $c_2(\theta, \varphi) \leqq c_2(\theta, \varphi^*)$ for all $\theta \in \Omega$. This completes the proof of the theorem.

THEOREM 2. *$\mathfrak{F}$ is essentially complete in $\mathfrak{M}$.*

PROOF. For a given W-type $m \in \mathfrak{M}$, put $\varphi_x(\theta) = m_x(S_\theta)$. Then by Lemma 2, we have $\varphi_x \in \Phi$ and $\int_{-\infty}^{\infty} \varphi_x(\eta) d\eta = \int_A (v-u) dm_x(u, v)$. It is clear φ satisfies (a′) and (b′) since m satisfies (a) and (b).

The above equality implies $w_2(\theta, \varphi) = w_1(\theta, m)$ and it is obvious $c_2(\theta, \varphi) = c_1(\theta, m)$ for all $\theta \in \Omega$. Then $\mathfrak{F}$ is essentially complete in $\mathfrak{M}$.

THEOREM 3. *$\mathfrak{M}^*$ is essentially complete in $\mathfrak{F}^*$.*

PROOF. For a given $\varphi^* \in \mathfrak{F}^*$, Lemma 3 shows that there exist monotone non-increasing functions $u_x(z)$ and $v_x(z)$ defined on $(0, 1)$ satisfying $\varphi_x^*(\theta) = \int_{u_x(z) \leqq \theta \leqq v_x(z)} dz$ and $\int_{-\infty}^{\infty} \varphi_x^*(\eta) d\eta = \int_0^1 (v_x(z) - u_x(z)) dz$.

Putting $m_x(S) = \int_{S \ni (u_x(z), v_x(z))} dz$, we have $m_x \in M$ for every x and by Lemmas 3 and 4, we obtain

$$\varphi_x^*(\theta) = m_x(S_\theta)$$

and

$$\int_{-\infty}^{\infty} \varphi_x^*(\eta)d\eta = \int_A (v-u)\, dm_x(u,v)\ .$$

Above two equalities imply that $m \in \mathfrak{M}^*$ and $w_1(\theta, m) = w_2(\theta, \varphi^*)$, $c_1(\theta, m) = c_2(\theta, \varphi^*)$ for all $\theta \in \Omega$. $m \in \mathfrak{M}^*$ follows from the definition of m. Thus $\mathfrak{M}^*$ is essentially complete in $\mathfrak{F}^*$.

THEOREM 4. *$\mathfrak{M}^*$ is essentially complete in $\mathfrak{M}$.*

PROOF. This follows directly from preceding theorems.

THEOREM 5. *$\mathfrak{M}$, $\mathfrak{F}$, $\mathfrak{M}^*$ and $\mathfrak{F}^*$ are equivalent.*

PROOF. From above theorems, we can easily see

$$\mathfrak{M} > \mathfrak{F} > \mathfrak{F}^* > \mathfrak{M}^* > \mathfrak{M}\ .$$

The last relation is obvious since $\mathfrak{M}^* \subset \mathfrak{M}$. This chain of relations shows equivalences of $\mathfrak{M}$, $\mathfrak{M}^*$, $\mathfrak{F}$ and $\mathfrak{F}^*$.

REMARK 1. m in $\mathfrak{M}^*$ is obviously a randomized confidence interval defined by Fraser ([3]).

REMARK 2. For a given $\varphi \in \mathfrak{F}^*$, we can construct a different $m \in \mathfrak{M}$ from one considered in the Theorem 3 though they are quite same with w and c.

The author wishes to express his gratitude to Prof. H. Kudo for his suggestion and continuing encouragement in writing of this note.

References

[1] A. Wald, Statistical decision functions, New York, 1950.
[2] C. Stein, Some problems in multivariate analysis, (part 1) Technical Report no. 6, Stanford University, 1956.
[3] D. A. S. Fraser, Nonparametric methods in Statistics, New York, 1957, p. 111.